

# HOMOLOGY AND MODULAR CLASSES OF LIE ALGEBROIDS

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**ABSTRACT.** For a Lie algebroid, divergences chosen in a classical way lead to a uniquely defined homology theory. They define also, in a natural way, modular classes of certain Lie algebroid morphisms. This approach, applied for the anchor map, recovers the concept of modular class due to S. Evens, J.-H. Lu, and A. Weinstein.

## 1. INTRODUCTION

Homology of a Lie algebroid structure on a vector bundle  $E$  over  $M$  are usually considered as homology of the corresponding Batalin-Vilkovisky algebra associated with a chosen generating operator  $\partial$  for the Schouten-Nijenhuis bracket on multi-sections of  $E$ . The generating operators that are homology operators, i.e.  $\partial^2 = 0$ , can be identified with flat  $E$ -connections on  $\bigwedge^{\text{top}} E$  (see [17]) or divergence operators (flat right  $E$ -connections on  $M \times \mathbb{R}$ , see [8]). The problem is that the homology group depends on the choice of the generating operator (flat connection, divergence) and no one seems to be privileged. For instance, if a Lie algebroid on  $T^*M$  associated with a Poisson tensor  $P$  on  $M$  is concerned, then the traditional Poisson homology is defined in terms of the Koszul-Brylinski homology operator  $\partial_P = [d, i_P]$ . However, the Poisson homology groups may differ from the homology groups obtained by means of 1-densities on  $M$ . The celebrated modular class of the Poisson structure [18] measures this difference. Analogous statement is valid for triangular Lie bialgebroids [11].

The concept of a Lie algebroid divergence, so a generating operator, associated with a ‘volume form’, i.e. nowhere-vanishing section of  $\bigwedge^{\text{top}} E^*$ , is completely classical (see [11], [17]). Less-known seems to be the fact that we can use ‘odd-forms’ instead of forms (cf. [2]) with same formulas for divergence and that such nowhere-vanishing volume odd-forms always exist. The point is that the homology groups obtained in this way are all isomorphic, independently on the choice of the volume odd-form. This makes the homology of a Lie algebroid a well-defined notion. From this point of view the Poisson homology is not the homology of the associated Lie algebroid  $T^*M$  but a deformed version of the latter, exactly as the exterior differential  $d^\phi\mu = d\mu + \phi \wedge \mu$  of Witten [19] is a deformation of the standard de Rham differential.

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*Date:* February 1, 2008 .

*2000 Mathematics Subject Classification.* Primary 17B56, 17B66, 17B70; Secondary 53C05 .

*Key words and phrases.* Lie algebroid, de Rham cohomology, Poincaré duality, divergence.

The research of J. Grabowski supported by the Polish Ministry of Scientific Research and Information Technology under the grant No. 2 P03A 020 24. The research of P. Michor was supported by FWF Projects P 14195 and P 17108.

In this language, the modular class of a Lie algebroid morphism  $\kappa : E_1 \rightarrow E_2$  covering the identity on  $M$  is defined as the class of the difference between the pull-back of a divergence on  $E_2$  and a divergence on  $E_1$ , both associated with volume odd-forms. In the case when  $\kappa : E \rightarrow TM$  is the anchor map, we recognize the standard modular class of a Lie algebroid [3] but it is clear that other (canonical) morphisms will lead to other (canonical) modular classes.

## 2. DIVERGENCES AND GENERATING OPERATORS

**2.1. Lie algebroids and their cohomology.** Let  $\tau : E \rightarrow M$  be a vector bundle. Let  $\mathcal{A}^i(E) = \text{Sec}(\bigwedge^i E)$  for  $i = 0, 1, 2, \dots$ , let  $\mathcal{A}^i(E) = \{0\}$  for  $i < 0$ , and denote by  $\mathcal{A}(E) = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}^i(E)$  the Grassmann algebra of multisections of  $E$ . It is a graded commutative associative algebra with respect to the wedge product.

There are different ways to define a Lie algebroid structure on  $E$ . We prefer to see it as a linear graded Poisson structure on  $\mathcal{A}(E)$  (see [7]), i.e., a graded bilinear operation  $[ \ , \ ]$  on  $\mathcal{A}(E)$  of degree  $-1$  with the following properties:

- (a) Graded anticommutativity:  $[a, b] = -(-1)^{(|a|-1)(|b|-1)}[b, a]$ .
- (b) The graded Jacobi identity:  $[a, [b, c]] = [[a, b], c] + (-1)^{(|a|-1)(|b|-1)}[b, [a, c]]$ .
- (c) The graded Leibniz rule:  $[a, b \wedge c] = [a, b] \wedge c + (-1)^{(|a|-1)|b|}b \wedge [a, c]$ .

This bracket is just the Schouten bracket associated with the the standard Lie algebroid bracket on sections of  $E$ . It is well known that such brackets are in bijective correspondence with de Rham differentials  $d$  on the Grassmann algebra  $\mathcal{A}(E^*)$  of multisections of the dual bundle  $E^*$  which are described by the formula

$$(1) \quad d\mu(X_0, \dots, X_n) = \sum_i (-1)^i [X_i, \mu(X_0, \dots \hat{i} \dots, X_n)] + \\ + \sum_{k < l} (-1)^{k+l} \mu([X_k, X_l], X_0, \dots \hat{k} \dots \hat{l} \dots, X_n)$$

where the  $X_i$  are sections of  $E$ . We will refer to elements of  $\mathcal{A}(E^*)$  as *forms*. Since  $d$  is a derivation on  $\mathcal{A}(E^*)$  of degree 1 with  $d^2 = 0$ , it defines the corresponding de Rham cohomology  $H^*(E, d)$  of the Lie algebroid in the obvious way.

**2.2. Generating operators and divergences.** The definition of the homology of a Lie algebroid is more delicate than that of cohomology. The standard approach is via generating operators for the Schouten bracket  $[ \ , \ ]$ . By this we mean an operator  $\partial$  of degree  $-1$  on  $\mathcal{A}(E)$  which satisfies

$$(2) \quad [a, b] = (-1)^{|a|}(\partial(a \wedge b) - \partial(a) \wedge b - (-1)^{|a|}a \wedge \partial(b)).$$

The idea of a generating operator goes back to the work by Koszul [14]. A generating operator which is a homology operator, i.e.  $\partial^2 = 0$ , gives rise to the so called Batalin-Vilkovisky algebra. Remark that the leading sign  $(-1)^{|a|}$  serves to produce graded antisymmetry with respects to the degrees shifted by  $-1$  out of graded symmetry. One could equally well use  $(-1)^{|b|}$  instead of  $(-1)^{|a|}$ , or one could use the obstruction for  $\partial$  to be a graded right derivation in the parentheses instead of a graded left one as we did. We shall stick to the standard conventions.

It is clear from (2) and from the properties of the Schouten bracket that  $\partial$  is then a second order differential operator on the graded commutative associative

algebra  $\mathcal{A}(E)$  which is completely determined by its restriction to  $\text{Sec}(E)$ . In fact, it is easy to see (cf. [8]) that

$$(3) \quad \partial(X_1 \wedge \cdots \wedge X_n) = \sum_i (-1)^{i+1} \partial(X_i) X_1 \wedge \cdots \hat{i} \cdots \wedge X_n + \\ + \sum_{k < l} (-1)^{k+l} [X_k, X_l] \wedge X_1 \wedge \cdots \hat{k} \cdots \hat{l} \cdots \wedge X_n$$

for  $X_1, \dots, X_n \in \text{Sec}(E)$ , which looks completely dual to (1). From (2) we get the following property of  $\partial$ :

$$(4) \quad -\partial(fX) = -f\partial(X) + [X, f] \quad \text{for } X \in \text{Sec}(E), f \in C^\infty(M).$$

Since  $[X, f] = \rho(X)(f)$  where  $\rho : E \rightarrow TM$  is the anchor map of the Lie algebroid structure on  $E$ , the operator  $-\partial$  has the algebraic property of a divergence. Conversely, (3) defines a generating operator for  $[\cdot, \cdot]$  if only (4) is satisfied. i.e., generating operators can be identified with divergences. We may express this by  $\text{div} \leftrightarrow \partial_{\text{div}}$ . But a true divergence  $\text{div} : \text{Sec}(E) \rightarrow C^\infty(M)$  satisfies besides (4) a cocycle condition

$$(5) \quad \text{div}([X, Y]) = [\text{div}(X), Y] + [X, \text{div}(Y)], \quad X, Y \in \text{Sec}(E),$$

which is equivalent (see [8]) to the fact that the corresponding generating operator  $\partial_{\text{div}}$  is a homology operator:  $(\partial_{\text{div}})^2 = 0$ . Note that divergences can be used in construction of generating operators also in the supersymmetric case (cf. [13]).

From now on we will fix the Lie algebroid structure on  $E$ , and we will denote by  $\text{Gen}(E)$  the set of generating operators for  $[\cdot, \cdot]$  which are homology operators, and by  $\text{Div}(E)$  the canonically isomorphic (by (3)) set of divergences for the Lie algebroid satisfying (4) and (5). The problem is that there does not exist a canonical divergence, thus no canonical generating operator.

The set  $\text{Div}(E)$  can be identified with the set of all flat  $E$ -connections on  $\bigwedge^{\text{top}} E^*$ , i.e., operators  $\nabla : \text{Sec}(E) \times \text{Sec}(\bigwedge^{\text{top}}(E^*)) \rightarrow \text{Sec}(\bigwedge^{\text{top}}(E^*))$  which satisfy

- (i)  $\nabla_{fX}\mu = f\nabla_X\mu$ ,
- (ii)  $\nabla_X(f\mu) = f\nabla_X\mu + \rho(X)(f)\mu$ ,
- (iii)  $[\nabla_X, \nabla_Y] = \nabla_{[X, Y]}$ .

The identification is via

$$(6) \quad \mathcal{L}_X\mu - \nabla_X\mu = \text{div}(X)\mu$$

(cf. [11, (50)]), where  $\mathcal{L}_X = di_X + i_X d$  is the Lie derivative. Note that (6) is independent of the choice of the section  $\mu \in \text{Sec}(\bigwedge^{\text{top}}(E^*))$ . We can use  $\bigwedge^{\text{top}}(E)$  instead of  $\bigwedge^{\text{top}}(E^*)$  and get the identification of  $\text{Div}(E)$  with the set of flat  $E$ -connection on  $\bigwedge^{\text{top}}(E)$  by (see [17])

$$(7) \quad \mathcal{L}_X\Lambda - \nabla_X\Lambda = \text{div}(X)\Lambda.$$

Of course, additional structures on  $E$  as, e.g., a Riemannian metric (smoothly arranged scalar products on fibers of  $E$ ), may furnish a distinguished divergence on  $E$ . Fixing a metric we can distinguish a canonical torsionfree connection  $\nabla$  on  $E$  - the Levi-Civita connection for the Lie algebroid - in the standard way. It satisfies the standard Bianchi and Ricci identities (see [16]) and induces a connection on  $\bigwedge^{\text{top}}(E)$  for which the generating operator  $\partial_\nabla$  has the local form (see [17])

$\partial_\nabla(a) = -\sum_k i(\alpha^k) \nabla_{X_k} a$  where the  $X_k$  and  $\alpha^k$  are dual local frames for  $E$  and  $E^*$ , respectively. Since

$$\partial_\nabla^2 = \sum_{k,j} i(\alpha^j) \nabla_{X_j} i(\alpha^k) \nabla_{X_k} = \sum_{k,j} i(\alpha^j) i(\alpha^k) (\nabla_{X_j} \nabla_{X_k} - \nabla_{\nabla_{X_j} X_k}),$$

$\partial^2 = 0$  is equivalent to

$$(8) \quad \sum_{j,k} i(\alpha^j) i(\alpha^k) R(X_j, X_k) = 0,$$

where  $R$  is the curvature tensor of  $\nabla$ . For a Levi-Civita connection  $\nabla$  the generating operator  $\partial_\nabla$  is really a homology operator due to the following lemma.

**Lemma 2.3.** *A torsionfree connection  $\nabla$  on  $E$  satisfies simultaneously the Bianchi and the Ricci identity if and only if (8) holds for dual local frames  $X_k$  and  $\alpha^k$  of  $E$  and  $E^*$ , respectively.*

**Proof.** (8) is equivalent to  $\sum_{j,k} R(X_j, X_k)^*(\alpha^k \wedge \alpha^j \wedge \omega) = 0$  for all forms  $\omega$ . It suffices to check this for  $\omega$  a function or a 1-form due to the derivation property of contractions. For  $\omega$  a function  $f$  we have

$$\begin{aligned} \sum_{j,k} R(X_j, X_k)^*(f \alpha^k \wedge \alpha^j) &= \sum_{j,k} f \left( R(X_j, X_k)^*(\alpha^k) \wedge \alpha^j + \alpha^k \wedge R(X_j, X_k)^*(\alpha^j) \right) \\ &= 2f \sum_{s,j,k} R_{jks}^k \alpha^s \wedge \alpha^j \end{aligned}$$

and this vanishes for all  $f$  if and only if  $R_{jks}^k$  is symmetric in  $(j, s)$ , i.e., if the Ricci identity holds. For  $\omega$  a 1-form, say  $\alpha^i$ , we have

$$\begin{aligned} \sum_{j,k} R(X_j, X_k)^*(\alpha^k \wedge \alpha^j \wedge \alpha^i) &= \\ &= \sum_{j,k} \left( R(X_j, X_k)^*(\alpha^k \wedge \alpha^j) \wedge \alpha^i + \alpha^k \wedge \alpha^j \wedge R(X_j, X_k)^*(\alpha^i) \right) = \\ &= 0 + \sum_{j,k,s} R_{jks}^i \alpha^k \wedge \alpha^j \wedge \alpha^s \end{aligned}$$

and this vanishes for all  $i$  if and only if  $\sum_{\text{cycl}(j,k,s)} R_{jks}^i = 0$ , i.e., if the first Bianchi identity holds.  $\square$

**Corollary 2.4.** *Any Levi-Civita connection for a Riemannian metric on a Lie algebroid  $E$  induces a flat connection on  $\bigwedge^{\text{top}} E$ , thus also on  $\bigwedge^{\text{top}} E^*$ .*  $\square$

### 3. HOMOLOGY OF THE LIE ALGEBROID

**3.1. Getting divergences from odd forms.** There is no distinguished divergence for the Lie algebroid structure on  $E$ , but there is a distinguished subset of divergences which we may obtain in a classical way. Firstly, suppose that the line bundle  $\bigwedge^{\text{top}} E^*$  is trivializable. So we can choose a vector volume, i.e., a nowhere vanishing section  $\mu \in \text{Sec}(\bigwedge^{\text{top}} E^*)$ . Then the formula

$$(9) \quad \mathcal{L}_X \mu = \text{div}_\mu(X) \mu, \quad \text{where } X \in \text{Sec}(E)$$

defines a divergence  $\text{div}_\mu$ . We observe that  $\text{div}_{-\mu} = \text{div}_\mu$ . Thus for the non-orientable case we look for sections of a bundle over  $M$  which locally consists of non-ordered pairs  $\{\mu_\alpha, -\mu_\alpha\}$  for an open cover  $M = \bigcup_\alpha U_\alpha$  such that the sets  $\{\mu_\alpha, -\mu_\alpha\}$  and  $\{\mu_\beta, -\mu_\beta\}$  coincide when restricted to  $U_\alpha \cap U_\beta$ . The fundamental observation is that such global sections always exist and define global divergences. This is because they can be viewed as sections of the bundle  $|\text{Vol}|_E = (\bigwedge^{\text{top}} E^*)_0 / \mathbb{Z}_2$ , where  $(\bigwedge^{\text{top}} E^*)_0$  is the bundle  $\bigwedge^{\text{top}} E^*$  with the zero section removed and divided by the obvious  $\mathbb{Z}_2$ -action of passing to the opposite vector. The bundle  $|\text{Vol}|_E$  is an 1-dimensional affine bundle modelled on the vector bundle  $M \times \mathbb{R}$ , and also a principal  $\mathbb{R}$  bundle where  $t \in \mathbb{R}$  acts by scalar multiplication with  $e^t$ . Since it has a contractible fiber, sections always exist. Note that sections  $|\mu|$  of  $|\text{Vol}|_E$  are particular cases of odd forms, [2]: Let  $p: \tilde{M} \rightarrow M$  be the two-fold covering of  $M$  on which  $p^*E$  is oriented, namely the set of vectors of length 1 in the line bundle over  $M$  with cocycle of transition functions  $\text{sign} \det(\phi_{\alpha\beta})$ , where  $\phi_{\alpha\beta}: U_\alpha \cap U_\beta \rightarrow GL(V)$  is the cocycle of transition functions for the vector bundle  $E$ . Then the odd forms are those forms on  $p^*E$  which are in the  $-1$  eigenspace of the natural vector bundle isomorphism which covers the decktransformation of  $\tilde{M}$ . So odd forms are certain sections of a line bundle over a two-fold covering of the base manifold  $M$ . This is related but complementary to the construction of the line bundle (over  $M$ ) of densities which involve the cocycle of transition functions  $|\det(\phi_{\alpha\beta})|$ . For example, any Riemannian metric  $g$  on the vector bundle  $E$  induces an odd volume form  $|\mu|_g \in \text{Sec}(|\text{Vol}|_E) \simeq \text{Sec}(|\text{Vol}|_{E^*})$  which locally is represented by the wedge product of any orthonormal basis of local sections of  $E$  (thus  $E^*$ ). Note that such product is independent on the choice of the basis modulo sign, so our odd volume is well defined.

For the definition of a divergence  $\text{div}_{|\mu|}$  associated to  $|\mu| \in \text{Sec}(|\text{Vol}|_E)$  we will write simply

$$(10) \quad \mathcal{L}_X |\mu| = \text{div}_{|\mu|}(X)|\mu| \quad \text{for } X \in \text{Sec}(E).$$

Note that the distinguished set  $\text{Div}_0(E)$  of divergences obtained in this way from sections of  $|\text{Vol}|_E$  corresponds (in the sense of (6)) to the set of those flat connections on  $\bigwedge^{\text{top}} E^*$  whose holonomy group equals  $\mathbb{Z}_2$ : Associate the horizontal leaf  $|\mu|$  to such a connection, and note that a positive multiple of  $|\mu|$  gives rise to the same divergence.

In the case of a vector bundle Riemannian metric  $g$  on  $E$  a natural question arises about the relation between the divergence  $\text{div}_{|\mu|_g}$  associated with the odd volume  $|\mu|_g$  induced by the metric  $g$  and the divergence  $\text{div}_{\nabla_g}$  induced by the flat Levi-Civita connection  $\nabla_g$  on  $\bigwedge^{\text{top}} E^* \simeq \bigwedge^{\text{top}} E$ .

**Theorem 3.2.** *For any vector bundle Riemannian metric  $g$  on  $E$*

$$\text{div}_{|\mu|_g} = \text{div}_{\nabla_g}.$$

**Proof.** Let  $X_1, \dots, X_n$  be an orthonormal basis of local sections of  $E$  and  $\alpha^k = g(X_k, \cdot)$  be the dual basis of local sections of  $E^*$ , so that  $|\mu|_g$  is locally

represented by  $\alpha_1 \wedge \dots \wedge \alpha^n$ . For any local section  $X$  of  $E$

$$\begin{aligned} \text{div}_{|\mu|_g}(X) &= -\langle \mathcal{L}_X(\alpha^1 \wedge \dots \wedge \alpha^n), X_1 \wedge \dots \wedge X_n \rangle = \\ &\langle \alpha^1 \wedge \dots \wedge \alpha^n, \mathcal{L}_X(X_1 \wedge \dots \wedge X_n) \rangle = \sum_k \langle \alpha^k, [X, X_k] \rangle = \\ &\sum_k \langle \alpha^k, \nabla_X X_k - \nabla_{X_k} X \rangle = \sum_k g(X_k, \nabla_X X_k) - \sum_k i(\alpha^k) \nabla_{X_k} X. \end{aligned}$$

But  $-\sum_k i(\alpha^k) \nabla_{X_k} X = \text{div}_{\nabla_g}(X)$  and

$$2 \sum_k g(X_k, \nabla_X X_k) = \sum_k \rho(X) g(X_k, X_k) - \sum_k \nabla_X(g)(X_k, X_k) = 0,$$

where  $\rho : E \rightarrow TM$  is the anchor of the Lie algebroid on  $E$ , since  $\nabla$  is Levi-Civita ( $\nabla g = 0$ ).  $\square$

**3.3. The generating operator for an odd form.** The corresponding generating operator  $\partial_{|\mu|}$  for the divergence of a non-vanishing odd form  $|\mu|$  can be defined explicitly by

$$\mathcal{L}_a |\mu| = -i(\partial_{|\mu|}(a)) |\mu|,$$

where  $\mathcal{L}_a = i_a d - (-1)^{|a|} d i_a$  is the Lie differential associated with  $a \in \mathcal{A}^{|a|}(E)$  so that

$$(11) \quad i(\partial_{|\mu|}(a)) |\mu| = (-1)^{|a|} d i_a |\mu|.$$

In other words, locally over  $U$  we have

$$(12) \quad \partial_{|\mu|}(a) = (-1)^{|a|} *_{\mu}^{-1} d *_{\mu}(a)$$

where  $*_{\mu}$  is the isomorphism of  $\mathcal{A}(E)|_U$  and  $\mathcal{A}(E^*)|_U$  given by  $*_{\mu}(a) = i_a \mu$ , for a representative  $\mu$  of  $|\mu|$ . Note that the right hand side of (12) depends only on  $|\mu|$  and not on the choice of the representative since  $*_{\mu} d *_{\mu} = *_{-\mu} d *_{-\mu}$ . Formula (12) gives immediately  $\partial_{|\mu|}^2 = 0$ , which also follows from the remark on flat connections above. So  $\partial_{|\mu|}$  is a homology operator.

Moreover, it is also a generating operator. Namely, using standard calculus of Lie derivatives we get

$$\mathcal{L}_{a \wedge b} = i_b \mathcal{L}_a - (-1)^{|a|} i_{[a,b]} + (-1)^{|a| |b|} i_a \mathcal{L}_b$$

which can be rewritten in the form

$$(13) \quad i_{[a,b]} = (-1)^{|a|} \left( -\mathcal{L}_{a \wedge b} + i_b \mathcal{L}_a + (-1)^{|a|(|b|+1)} i_a \mathcal{L}_b \right)$$

When we apply (13) to  $|\mu|$  we get

$$i_{[a,b]} |\mu| = (-1)^{|a|} \left( i(\partial_{|\mu|}(a \wedge b)) - i(\partial_{|\mu|}(a) \wedge b) - (-1)^{|a|} i(a \wedge \partial_{|\mu|}(b)) \right) |\mu|$$

which proves (2). Thus we get:

**Theorem 3.4.** *For any  $|\mu| \in \text{Sec}(|\text{Vol}|_E)$  the formula*

$$(14) \quad \mathcal{L}_a |\mu| = -i(\partial_{|\mu|}(a)) |\mu|$$

*defines uniquely a generating operator  $\partial_{|\mu|} \in \text{Gen}(E)$ .*

We remark that formula (14) in the case of trivializable  $\bigwedge^{\text{top}} E^*$  has been already found in [11]. In this sense the formula is well known. What is stated in Theorem 3.4 is that (14) serves in general, as if the bundle  $\bigwedge^{\text{top}} E^*$  were trivial, if we replace ordinary forms with odd volume forms.

**3.5. Homology of the Lie algebroid.** The homology operator of the form  $\partial_{|\mu|}$  will be called the homology operator for the Lie algebroid  $E$ . The crucial point is that they all define the same homology. This is due to the fact that  $\partial_{|\mu_1|}$  and  $\partial_{|\mu_2|}$  differ by contraction with an exact 1-form.

In general, two divergences differ by contraction with a closed 1-form. Indeed,  $(\text{div}_1 - \text{div}_2)(fX) = f(\text{div}_1 - \text{div}_2)(X)$ , so  $(\text{div}_1 - \text{div}_2)(X) = i_\phi X$  for a unique 1-form  $\phi$ . Moreover, (5) implies that  $i_\phi[X, Y] = [i_\phi X, Y] + [X, i_\phi Y]$ , so  $\phi$  is closed. Since both sides are derivations we have

$$(15) \quad \partial_{\text{div}_2} - \partial_{\text{div}_1} = i_\phi.$$

But for any  $|\mu_1|, |\mu_2| \in \text{Sec}(|\text{Vol}|_E)$  there exists a positive function  $F = e^f$  such that  $|\mu_2| = F|\mu_1|$ . Then  $\mathcal{L}_X|\mu_2| = \mathcal{L}_X(F|\mu_1|) = \mathcal{L}_X(F)|\mu_1| + F\mathcal{L}_X(|\mu_1|)$  so that  $\text{div}_{|\mu_2|}(X)|\mu_2| = \mathcal{L}_X(f)|\mu_2| + \text{div}_{|\mu_1|}(X)|\mu_2|$ , i.e.,  $\text{div}_{|\mu_2|} - \text{div}_{|\mu_1|} = i(df)$ .

To see that the homology of  $\partial_{|\mu_1|}$  and  $\partial_{|\mu_2|}$  are the same, note first that  $\partial_{|\mu_2|} = \partial_{|\mu_1|}a + i_{df}a$ . And then let us gauge  $\mathcal{A}(E)$  by multiplication with  $F = e^f$ . This is an isomorphism of graded vector spaces and we have

$$e^f \partial_{|\mu_1|} e^{-f} a = \partial_{|\mu_1|}a + i_{df}a = \partial_{|\mu_2|}a,$$

so  $\partial_{|\mu_1|}$  and  $\partial_{|\mu_2|}$  are graded conjugate operators.

This is just the dual picture of the well-known gauging of the de Rham differential by Witten [19], see also [7] for consequences in the theory of Lie algebroids. Thus we have proved (cf. [11, p.120]):

**Theorem 3.6.** *All homology operators for a Lie algebroid generate the the same homology:  $H_*(E, \partial_{|\mu_1|}) = H_*(E, \partial_{|\mu_2|})$ . In the case of trivializable  $\bigwedge^{\text{top}} E^*$ , (12) gives Poincaré duality  $H^*(E, d) \cong H_{\text{top}-*}(E, \partial_{|\mu|})$ .*

**3.7. Remark.** We got a well-defined Lie algebroid homology, in contrast with the standard approach when all generating operators are admitted. It is clear that adding a term  $i_\phi$  with  $\phi$  a closed 1-form which is not exact, as in (15), will probably change the homology. But this could be understood as an a priori deformation like in the case of the deformed de Rham differential of Witten [19]:

$$(16) \quad d^\phi \eta = d\eta + \phi \wedge \eta.$$

Indeed,  $i(i_\phi a)\mu = -(-1)^{|a|}\phi \wedge i_a \mu$  implies  $*_\mu i_\phi(a) = -(-1)^{|a|}e_\phi *_\mu(a)$  where  $e_\phi \eta = \phi \wedge \eta$ . Thus we get  $(-1)^{|a|} *_\mu^{-1}(d + e_\phi) *_\mu(a) = (\partial_{|\mu|} - i_\phi)(a)$ , so, at least in the the trivializable case, there is the Poincaré duality

$$H^*(E, d + e_\phi) \cong H_{\text{top}-*}(E, \partial_\mu - i_\phi).$$

Note that the differentials  $d^\phi$  appear as part of the Cartan differential calculus for Jacobi algebroids, see [10], [6], [7], so that there is a relation between generating operators for a Lie algebroid and the Jacobi algebroid structures associated with it.

## 4. MODULAR CLASSES

**4.1. The modular class of a morphism.** As we have shown, every Lie algebroid  $E$  has a distinguished class  $\text{Div}_0(E)$  of divergences obtained from sections of  $|\text{Vol}|_E$ . Such divergences differ by contraction with an exact 1-form. Let now  $\kappa : E_1 \rightarrow E_2$  be a morphism of Lie algebroids.

There is the induced map  $\kappa^* : \text{Div}(E_2) \rightarrow \text{Div}(E_1)$  defined by  $\kappa^*(\text{div}_2)(X_1) = \text{div}_2(\kappa(X_1))$ . The fact that  $\kappa^*$  maps divergences into divergences follows from  $\kappa(fX) = f\kappa(X)$  and the fact that the Lie algebroid morphism respects the anchors,  $\rho_1 = \rho_2 \circ \kappa$ . The space  $\kappa^*(\text{Div}_0(E_2)) \subset \text{Div}(E_1)$  consists of divergences which differ by insertion of an exact 1-form. Therefore, the cohomology class of the 1-form  $\phi$  which is defined by the equation

$$(17) \quad \kappa^*(\text{div}_{E_2}) - \text{div}_{E_1} = i_\phi, \quad \text{for } \text{div}_{E_i} \in \text{Div}_0(E_i), i = 1, 2,$$

does not depend on the choice of  $\text{div}_{E_1}$  and  $\text{div}_{E_2}$ . We will call it the *modular class* of  $\kappa$  and denote it by  $\text{Mod}(\kappa)$ . Thus we have:

**Theorem 4.2.** *For every Lie algebroid morphism*

$$\begin{array}{ccc} E_1 & \xrightarrow{\kappa} & E_2 \\ & \searrow \tau_1 & \swarrow \tau_2 \\ & M & \end{array}$$

the cohomology class  $\text{Mod}(\kappa) = [\phi] \in H^1(E_1, d_{E_1})$  defined by  $\phi$  in (17) is well defined independently of the choice of  $\text{div}_{E_1} \in \text{Div}_0(E_1)$  and  $\text{div}_{E_2} \in \text{Div}_0(E_2)$ .  $\square$

**4.3. The modular class of a Lie algebroid.** In the case when the morphism  $\kappa = \rho : E \rightarrow TM$  is the anchor map of a Lie algebroid  $E$ , the modular class  $\text{Mod}(\rho)$  is called the *modular class of the Lie algebroid*  $E$  and it is denoted by  $\text{Mod}(E)$ . The idea that the modular class is associated with the difference between the Lie derivative action on  $\bigwedge^{\text{top}}(E^*)$  and on  $\bigwedge^{\text{top}}T^*M$  via the anchor map is, in fact, already present in [3]. Also the interpretation of the modular class as certain secondary characteristic class of a Lie algebroid, present in [4], is a quite similar. In [4] the trace of the difference of some connections is used instead of the difference of two divergences. We have

**Theorem 4.4.**  *$\text{Mod}(E)$  is the modular class  $\Theta_E$  in the sense of [3].*

**Proof.** The modular class  $\Theta_E$  in the sense of [3] is defined as the class  $[\phi]$  where  $\phi$  is given by

$$(18) \quad \mathcal{L}_X(a) \otimes \mu + a \otimes \mathcal{L}_{\rho(X)}\mu = \langle X, \phi \rangle a \otimes \mu$$

for all sections  $a$  of  $\bigwedge^{\text{top}}(E)$  and  $\mu$  of  $\bigwedge^{\text{top}}(T^*M)$ , respectively. Let us take  $|a^*| \in \text{Sec}(|\text{Vol}|_E)$  and  $|\mu| \in \text{Sec}(|\text{Vol}|_{TM})$ , locally represented by  $a^* \in \text{Sec}(\bigwedge^{\text{top}}(E^*|_U))$  and  $\mu \in \text{Sec}(\bigwedge^{\text{top}}(T^*M|_U))$ . Let  $a$  be a local section of  $\bigwedge^{\text{top}}E$  dual to  $a^*$ . Then  $\mathcal{L}_X(a) = -\text{div}_{|a^*|}(X)a$  and  $\mathcal{L}_X(\mu) = \rho^*(\text{div}_{|\mu|})(X)\mu$  so that (18) yields  $i_\phi = \rho^*(\text{div}_{|\mu|}) - \text{div}_{|a^*|}$ .  $\square$

Note that in our approach the modular class  $\text{Mod}(TM)$  of the canonical Lie algebroid  $TM$  is trivial by definition. It is easy to see that the modular class of a base preserving morphism can be expressed in terms of the modular classes of the corresponding Lie algebroids.

**Theorem 4.5.** *For a base preserving morphism  $\kappa : E_1 \rightarrow E_2$  of Lie algebroids*

$$\text{Mod}(\kappa) = \text{Mod}(E_1) - \kappa^*(\text{Mod}(E_2)).$$

**Proof.** Let  $\rho_l : E_l \rightarrow TM$  be the anchor of  $E_l$ ,  $l = 1, 2$ . Take  $\text{div}_{E_l} \in \text{Div}_0(E_l)$ ,  $l = 1, 2$ , and  $\text{div}_{TM} \in \text{Div}_0(TM)$ . Since  $\text{Mod}(E_l)$  is represented by  $\eta_l$ ,  $i_{\eta_l} = \text{div}_{E_l} - \rho_l^*(\text{div}_{TM})$  and  $\rho_1 = \rho_2 \circ \kappa$ , we can write

$$\begin{aligned} i_{\eta_1} &= \text{div}_{E_1} - \rho_1^*(\text{div}_{TM}) = \\ &\quad \text{div}_{E_1} - \kappa^*(\text{div}_{E_2}) + \kappa^*(\text{div}_{E_2} - \rho_2^*(\text{div}_{TM})) = i_{\eta_\kappa} + i_{\kappa^*(\eta_2)}, \end{aligned}$$

where  $\eta_\kappa$  represents  $\text{Mod}(\kappa)$ . Thus  $\eta_1 = \eta_\kappa + \eta_2$ .  $\square$

**4.6. The universal Lie algebroid.** For any vector bundle  $\tau : E \rightarrow M$  there exists a universal Lie algebroid  $\text{QD}(E)$  whose sections are the quasi-derivations on  $E$ , i.e., mappings  $D : \text{Sec}(E) \rightarrow \text{Sec}(E)$  such that  $D(fX) = fD(X) + \hat{D}(f)X$  for  $f \in C^\infty(M)$  and  $X \in \text{Sec}(E)$ , where  $\hat{D}$  is a vector field on  $M$ ; see the survey article [5]. Quasi-derivations are known in the literature under various names: covariant differential operators [15], module derivations [16], derivative endomorphisms [12], etc. The Lie algebroid  $\text{QD}(E)$  can be described as the Atiyah algebroid associated with the principal  $GL(n, \mathbb{R})$ -bundle  $\text{Fr}(E)$  of frames in  $E$ , and quasi-derivations can be identified with the  $GL(n, \mathbb{R})$ -invariant vector fields on  $\text{Fr}(E)$ . The corresponding short exact Atiyah sequence in this case is

$$0 \rightarrow \text{End}(E) \rightarrow \text{QD}(E) \rightarrow TM \rightarrow 0.$$

This observation shows that there is a modular class associated to every vector bundle  $E$ , namely the modular class  $\text{Mod}(\text{QD}(E))$ , which is a vector bundle invariant.

It is also obvious that, viewing a flat  $E_0$ -connection (representation) in a vector bundle  $E$  over  $M$  for a Lie algebroid  $E_0$  over  $M$  as a Lie algebroid morphism  $\nabla : E_0 \rightarrow \text{QD}(E)$ , one can define the modular class  $\text{Mod}(\nabla)$ .

**Question.** *How is  $\text{Mod}(\text{QD}(E))$  related to other invariants of  $E$  (e.g. characteristic classes)?*

**4.7. Remark.** One can interpret the modular class  $\text{Mod}(E)$  of the Lie algebroid  $E$  as a "trace" of the adjoint representation. Indeed, if we fix local coordinates  $u^a$  on  $U \subset M$  a local frame  $X_i$  of local sections of  $E$  over  $U$ , and the dual frame  $\alpha^i$  of  $E^*$ , then the Lie algebroid structure is encoded in the "structure functions"

$$[X_i, X_j] = \sum_k c_{ij}^k X_k, \quad \rho(X_i) = \sum_a \rho_i^a \partial_{u^a}.$$

**Proposition.** *The modular class  $\text{Mod}(E)$  is locally represented by the closed 1-form*

$$(19) \quad \phi = \sum_i \left( \sum_k c_{ik}^k + \sum_a \frac{\partial \rho_i^a}{\partial u^a} \right) \alpha^i.$$

**Proof.** We insert into (18) the elements  $a = X_1 \wedge \dots \wedge X_n$  and  $\mu = du^1 \wedge \dots \wedge du^m$ . Since

$$\mathcal{L}_{X_i} a = \sum_k c_{ik}^k a \quad \text{and} \quad \mathcal{L}_{X_i} \mu = \sum_a \frac{\partial \rho_i^a}{\partial u^a} \mu,$$

we get

$$\langle X_i, \phi \rangle a \otimes \mu = \left( \sum_k c_{ik}^k + \sum_a \frac{\partial \rho_i^a}{\partial u^a} \right) a \otimes \mu. \quad \square$$

One could say that representing cohomology locally does not make much sense, e.g. the modular class  $\text{Mod}(TM)$  is trivial so locally trivial. However, remember that for a general Lie algebroid the Poincaré lemma does not hold: closed forms need not be locally exact. In particular, for a Lie algebra (with structure constants), (19) says that the modular class is just the trace of the adjoint representation. In any case, (19) gives us a closed form, which is not obvious on first sight. If  $E$  is a trivial bundle, (19) gives us a globally defined modular class in local coordinates.

**4.8. Remark.** As we have already mentioned, the modular class of a Lie algebroid is the first characteristic class of R. L. Fernandes [4]. There are also higher classes, shown in [1] to be characteristic classes of the anchor map, interpreted as a representation "up to homotopy". It is interesting if our idea can be adapted to describe these higher characteristic classes as well.

## 5. ACKNOWLEDGEMENT

The authors are grateful to Marius Crainic and Yvette Kosmann-Schwarzbach for providing helpful comments on the first version of this note.

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